Advanced Studies in Contemporary Mathematics
27 (2017), No. 1, pp. 125 - 130

www.jangjeon.or.kr
http://dx.doi.org/10.23001/ascm2017.27.1.125

THE MINIMUM MULTIPLICITY OF STAR GRAPHS

JEONG-HWAN LEE*, SUNG-YUN YANG**, AND SANG-MOK KIM †

Abstract. In 1972, the Ramsey number $r(S_\alpha, S_\beta)$ with respect to star graphs was given by Harary, where S_t denotes the star graph of order $t+1$. While this number says just the existence of S_α, or $\overline{S}_\beta$, the compliment of S_β, in every graph G of order $r(S_\alpha, S_\beta)$, it does not exactly say anything about the number of S_α, and $\overline{S}_\beta$ in the graph. In this paper, we first enumerate the total number of star graphs of order t and the complement graphs of star graphs of order t along with the degree sequence of G, denoted by $\sigma_t(G)$. Next, we determine the minimum number $\sigma_t(\mathcal{G}_n)$ among $\sigma_t(G)$'s for $G \in \mathcal{G}_n$, where $\mathcal{G}_n$ is the set of all graphs of order n. Finally, as a corollary, we construct a 'minimal graph' of order $r(S_t, S_t)$ in which there are $\sigma_t(\mathcal{G}_n)$ star graphs S_t's and the compliments of star graphs $\overline{S}_t$'s.

2000 Mathematics Subject Classification. 05C55.

Keywords and phrases. ramsey number, minimum multiplicity.

1. Introduction

For graphs G_1, and G_2, the Ramsey number with respect to G_1, and G_2, denoted by $r(G_1, G_2)$ is defined as the least integer n such that every graph G of order n contains G_1, or $\overline{G}_2$, the complements of G_2. The Ramsey number with respect to star graphs $r(S_\alpha, S_\beta)$ is given by Harary [2], as follows;

$$r(S_\alpha, S_\beta) = \begin{cases} \alpha+\beta-1 & \text{if } \alpha, \text{ and } \beta \text{ are both even.} \\ \alpha+\beta & \text{otherwise,} \end{cases}$$

where S_t denotes the star graph of order $t+1$, that is the complete bipartite graph $K_{1,t}$.

Let $\mathcal{G}_n$ be the set of all graphs of order n. For a star graph S_t, an internal vertex v of S_t is a vertex with $\deg v = t$. We use the notations $s_t(G)$, $\overline{s}_t(G)$, and $s_t(v, G)$ to denote the number of S_t in G, the number of $\overline{S}_t$ in G, and the number of S_t containing a vertex v, respectively. Define $\sigma_t(G) = s_t(G) + \overline{s}_t(G)$, called the S_t-multiplicity of G. Lastly, we define the minimum S_t-multiplicity of order n, as follows;

$$\sigma_t(\mathcal{G}_n) = \min_{G \in \mathcal{G}_n} \{\sigma_t(G)\}.$$

In 1959, A. W. Goodman [1] enumerated the total number of K_3 and $\overline{K}_3$ for a given degree sequence of a graph G, and determined their minimum for a given order. In this paper, we address the analogous question on the total number of star graphs and the complements of star graphs in a graph, and the minimum S_t-multiplicity among the graphs of order of n. The exact values of $\sigma_t(\mathcal{G}_n)$ are determined for the

† the corresponding author.
∗, ∗∗ undergraduate students.

cases of n modulo 4, respectively. As a corollary of this result, we finally have the minimum S_t-multiplicity $\sigma_t(\mathcal{G}_r)$ for $r = r(S_t, S_t)$, as follows.

$$s_t(G) + s_t(\overline{G}) \geq \sigma_t(\mathcal{G}_r) = \begin{cases} 1 & \text{if } t \text{ is even} \\ 2t & \text{if } t \text{ is odd} \end{cases}$$

Furthermore, we construct a 'minimal graph' of order $r(S_t, S_t)$ with the minimum S_t-multiplicity $\sigma_t(\mathcal{G}_n)$.

2. Counting properties of star graphs

Let G be a graph and $v \in V(G)$. For a positive integer t, we have $s_t(v, G) = \binom{\deg v}{t}$ since S_t consists of an internal vertex v, and t vertices adjacent to v. Now we count $\sigma_t(G)$ for a graph G, as follows.

Lemma 1. *Let G be a graph. Then*

$$\sigma_t(G) = \sum_{v \in G} \binom{\deg v}{t} + \sum_{v \in G} \binom{\overline{\deg}\ v}{t},$$

where $\overline{\deg}\ v$ is the number of non-adjacent vertices to v.

Proof. Note that

$$s_t(G) = \sum_{v \in G} s_t(v, G) = \sum_{v \in G} \binom{\deg v}{t}, \text{ and}$$
$$s_t(\overline{G}) = \sum_{v \in \overline{G}} s_t(v, \overline{G}) = \sum_{v \in \overline{G}} \binom{\deg v}{t} = \sum_{v \in G} \binom{\overline{\deg}\ v}{t}.$$

Since $\sigma_t(G) = s_t(G) + \overline{s}_t(G) = s_t(G) + s_t(\overline{G})$, the statement holds. □

In order to evaluate minimum of $\sigma_t(G)$ with graphs G of order n, which is $\sigma_t(\mathcal{G}_n)$, we give the following inequality standing for combination of integers.

Lemma 2. *Let $a, b, t \in \mathbb{N}$ with $a \geq b \geq t$, and let $a - b = k$. Then*

$$\binom{a}{t} + \binom{b}{t} \geq \binom{a - \lfloor \frac{k}{2} \rfloor}{t} + \binom{b + \lfloor \frac{k}{2} \rfloor}{t}$$

where the equality holds when $t, k \in \{0, 1\}$.

Proof. If $t, k \in \{0, 1\}$, then the equality holds. Let $t \geq 2$, $k \geq 2$ and $a - b = k$. Note that for $k \geq 2$

$$\binom{a}{t} + \binom{b}{t} > \binom{a-1}{t} + \binom{b+1}{t}.$$

By applying this process $\lfloor \frac{k}{2} \rfloor$ times, we have

$$\binom{a}{t} + \binom{b}{t} > \binom{a - \lfloor \frac{k}{2} \rfloor}{t} + \binom{b + \lfloor \frac{k}{2} \rfloor}{t}.$$

□

Now, we give two distinct constructions of some graphs with certain order conditions which will be used for the case when $\sigma_t(G) = \sigma_t(\mathcal{G}_n)$ in Theorem 7. It is well-known that the circulant graph $\mathrm{Circ}(n;X)$ is defined as follows (see [3]). For a positive integer n and a subset X of the set of integers $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, called the connections,

- $V(\mathrm{Circ}(n;X))$ is $\mathbb{Z}_n$, and
- $ij \in E(\mathrm{Circ}(n;X))$ if and only if $|i-j| \in X$.

By using this method to construct circulant graphs, the following Construction 3 gives two k-regular graphs of order n for integer k with $0 \le k \le n-1$.

Construction 3. *Let n and k be positive integers with $0 \le k \le n-1$.*

(1) If k is even, then set $X = \{1, 2, \ldots, \frac{k}{2}\}$, and we have Circ$(n;X)$, which is a k-regular graph of order n.

(2) If n is even and k is odd, then set $X = \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor, \frac{n}{2}\}$ and we have Circ$(n;X)$, which is a k-regular graph of order n.

We remark that, from Construction 3, we always have a k-regular graph of order n when nk is an even integer.

Construction 4. *We construct $G = (V,E)$ of order $4u+3$, as follows. V is defined as the union of mutually disjoint sets V_1, V_2, W_1 and W_2 such that $|V_1| = 1$, $|V_2| = 2u$, $|W_1| = 2$, and $|W_2| = 2u$. Next, we define the edge set E, as follows. (See example 5)*

(1) If $v \in V_1$, $w \in V_2$, then $vw \in E$.

(2) If $v \in V_1$, $w \in W_1$ or W_2, then $vw \notin E$.

(3) If $v, w \in V_2$, then $vw \in E$ for $v \neq w$.

(4) If $v \in V_2$, $w \in W_1$, then $vw \notin E$.

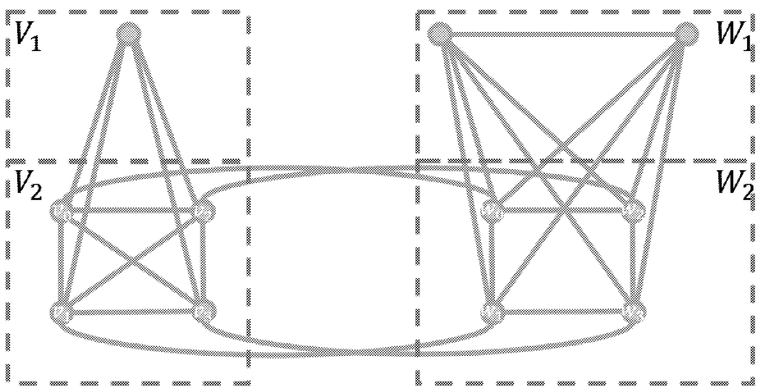
(5) Let $V_2 = \{v_0, v_1, \ldots, v_{2u-1}\}$, $W_2 = \{w_0, w_1, \ldots, w_{2u-1}\}$. For $v_i \in V_2$ and $w_j \in W_2$, $v_iw_j \in E$ if $i = j$. Otherwise, $v_iw_j \notin E$.

(6) If $v \in W_1$, $w \in W_1$, then $vw \in E$.

(7) If $v \in W_1$, $w \in W_2$, then $vw \in E$.

(8) Let $w_i \in W_2$ be labeled i with $i = 0, 1, \ldots, 2u-1$. Each vertex w_i is adjacent to vertices $i \pm 1, i \pm 2, \ldots, i \pm (u-1) \pmod{2u}$.

Example 5. *The graph of order 11 from Construction 4 is given as follows;*



Now, we give some properties of the graph constructed from Construction 4, as follows.

Proposition 6. *The graph G from Construction 4 satisfies the following degree properties.*

(i) $\deg v = 2u$ *for* $v \in V_1$.

(ii) $\deg v = 2u+1$ *for all* $v \in V \setminus V_1$.

Proof. Let v_1, v_2, w_1, and w_2 be some vertices in V_1, V_2, W_1, and W_2, respectively. Then, we have

(i) $\deg v_1 = |\{v \in V(G) \mid v_1 v \in E\}| = |\{v \in V_2 \mid v_1 v \in E\}| = 2u.$

(ii) $$\begin{aligned}\deg v_2 &= |\{v \in V(G) \mid v_2 v \in E\}| \\ &= |\{v \in V_1 \mid v_2 v \in E\}| + |\{v \in V_2 \mid v_2 v \in E\}| + |\{v \in W_2 \mid v_2 v \in E\}| \\ &= 1 + (2u-1) + 1 = 2u+1.\end{aligned}$$

(iii) $$\begin{aligned}\deg w_1 &= |\{v \in V(G) \mid w_1 v \in E\}| \\ &= |\{v \in W_1 \mid w_1 v \in E\}| + |\{v \in W_2 \mid w_1 v \in E\}| \\ &= 1 + 2u\end{aligned}$$

(iv) $$\begin{aligned}\deg w_2 &= |\{v \mid w_2 v \in E\}| \\ &= |\{v \in V_2 \mid w_2 v \in E\}| + |\{v \in W_1 \mid w_2 v \in E\}| + |\{v \in W_2 \mid w_2 v \in E\}| \\ &= 1 + 2 + (2u-2) = 2u+1\end{aligned}$$

□

3. Determination of $\sigma_t(\mathcal{G}_n)$

We now establish $\sigma_t(\mathcal{G}_n)$, as follow.

Theorem 7. *Let G be a graph of order $n \geq r(S_t, S_t)$.*

$$\sigma_t(\mathcal{G}_n) = \begin{cases} n\left[\binom{\frac{n}{2}}{t} + \binom{\frac{n}{2}-1}{t}\right] & \text{if } n = 2u \\ 2n\binom{\lfloor \frac{n}{2} \rfloor}{t} & \text{if } n = 4u+1 \\ 2(n-1)\binom{\lfloor \frac{n}{2} \rfloor}{t} + \binom{\lfloor \frac{n}{2} \rfloor + 1}{t} + \binom{\lfloor \frac{n}{2} \rfloor - 1}{t} & \text{if } n = 4u+3 \end{cases}$$

Proof. Let G be a graph of order n. Note that

$$\begin{aligned}\sigma_t(\mathcal{G}_n) &= \min_{G \in \mathcal{G}_n} \{\sigma_t(G)\} \\ &= \min\{s_t(G) + s_t(\overline{G}) \mid G \in \mathcal{G}_n\} \\ &= \min\left\{ \sum_{v \in G} \left[\binom{\deg v}{t} + \binom{\overline{\deg}\ v}{t}\right] \,\middle|\, G \in \mathcal{G}_n \right\}.\end{aligned}$$

We have the following result from lemma 2.

(1) For $n = 2u$, we have

$$\sigma_t(\mathcal{G}_n) \geq \sum_{i=1}^{2u} \left[\binom{u}{t} + \binom{u-1}{t}\right] = 2u\left[\binom{u}{t} + \binom{u-1}{t}\right],$$

where the equality holds when $\deg v = u-1$ or $\deg v = u$ for all $v \in G$. We notice that the lower bound is tight since there exists the u-regular graph of order $2u$ from Construction 3.

(2) For $n = 4u+1$, we have

$$\sigma_t(\mathcal{G}_n) \geq \sum_{i=1}^{4u+1} \left[\binom{2u}{t} + \binom{2u}{t}\right] = (8u+2)\binom{2u}{t}.$$

The equality holds when $\deg v = 2u$ for all $v \in V(G)$ since the $2u$-regular graph exists from Construction 3.

(3) For $n = 4u+3$, we have

$$\sigma_t(\mathcal{G}_n) \geq \sum_{i=1}^{4u+3} \left[\binom{2u+1}{t} + \binom{2u+1}{t}\right].$$

Note that $\deg v = 2u+1$ for all $v \in G$ so that $\sum_{v\in G} \deg v = (4u+3)(2u+1) = 2|E|$, which is a contradiction. Thus there are even number of vertices of odd degree. If a vertex has an even degree, then $\sigma_t(\mathcal{G}_n) = \sum_{i=1}^{4u+2} \left[\binom{2u+1}{t} + \binom{2u+1}{t}\right] + \binom{2u}{t} + \binom{2u+2}{t}$. Here, the equality holds when the degree sequence of G is as follows:

$$[2u+1, \ldots, 2u+1, 2u] \text{ or } [2u+2, 2u+1, \ldots, 2u+1].$$

Then, Construction 4 gives the graphs having these degree sequences.

□

Corollary 8. *Let G be a graph of order $n = r(S_t, S_t)$. Then*

$$s_t(G) + s_t(\overline{G}) \geq \sigma_t(\mathcal{G}_n) = \begin{cases} 1 & \text{if } t \text{ is even} \\ 2t & \text{if } t \text{ is odd.} \end{cases}$$

Proof. From Harary [2], note that

$$n = r(S_t, S_t) = \begin{cases} 2t-1 & \text{if } t \text{ is even} \\ 2t & \text{if } t \text{ is odd.} \end{cases}$$

(i) If t is even, then $n = 2t-1$ so that

$$\begin{aligned} \sigma_t(\mathcal{G}_n) &= 2(n-1)\binom{\lfloor \frac{n}{2} \rfloor}{t} + \binom{\lfloor \frac{n}{2} \rfloor + 1}{t} + \binom{\lfloor \frac{n}{2} \rfloor - 1}{t} \\ &= 2(2t-2)\binom{\lfloor \frac{2t-1}{2} \rfloor}{t} + \binom{\lfloor \frac{2t-1}{2} \rfloor + 1}{t} + \binom{\lfloor \frac{2t-1}{2} \rfloor - 1}{t} \\ &= 2(2t-2)\binom{t-1}{t} + \binom{t}{t} + \binom{t-2}{t} \\ &= 1. \end{aligned}$$

(ii) If t is odd, then, $n = 2t$ so that

$$\begin{aligned}\sigma_t(\mathcal{G}_n) &= n\left[\binom{\frac{n}{2}}{t} + \binom{\frac{n}{2}-1}{t}\right] \\ &= 2t\left[\binom{\frac{2t}{2}}{t} + \binom{\frac{2t}{2}-1}{t}\right] \\ &= 2t\left[\binom{t}{t} + \binom{t-1}{t}\right] \\ &= 2t.\end{aligned}$$

□

From Corollary 8, we obtain the total number of S_t and $\overline{S}_t$ in the minimal graph of order $r(S_t, S_t)$ given by Construction 3 and Construction 4, respectively.

Acknowledgement

The present research has been conducted by the Research Grant of Kwangwoon University in 2016.

References

[1] Adolph W. Goodman, On sets of acquaintances and strangers at any party. The American Mathematical Monthly 66-9 (1959) 778-783.
[2] F. Harary, Recent results on generalized Ramsey theory for graphs, Graph Theory and Applications, Springer Berlin Heidelberg, (1972) 125-138.
[3] Gross, Yellen, and Zhang, Handbook of graph theory, CRC Press. (2014) 23-24.

Jeong-Hwan Lee, Department of Mathematics, Kwangwoon University
E-mail address: hwani@kw.ac.kr

Sung-Yun Yang, Department of Mathematics, Kwangwoon University
E-mail address: kuturu@naver.com

Sang-Mok Kim, Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea
E-mail address: smkim@kw.ac.kr